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Lie algebroid foliations and $\mathcal{E}^1(M)$-Dirac structures

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Abstract
We prove some general results about the relation between the 1-cocycles of an arbitrary Lie algebroid A over M and the leaves of the Lie algebroid foliation on M associated with A. Using these results, we show that a $\mathcal{E}^1(M)$-Dirac structure L induces on every leaf F of its characteristic foliation a $\mathcal{E}^1(F)$-Dirac structure L_F, which comes from a precontact structure or from a locally conformal presymplectic structure on F. In addition, we prove that a Dirac structure $\tilde{L}$ on $M \times \mathbb{R}$ can be obtained from L and we discuss the relation between the leaves of the characteristic foliations of L and $\tilde{L}$.

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1. Introduction

The fundamental role that Poisson algebras play in Dirac's theory of constrained Hamiltonian systems is well known [5]. Two natural ways for Poisson algebras to arise from a manifold M are through Poisson or presymplectic structures on M. Both structures are examples of Dirac structures in the sense of Courant–Weinstein [2, 3]. A Dirac structure on a manifold M is a vector sub-bundle $\tilde{L}$ of $TM \oplus T^*M$ that is maximally isotropic under the natural symmetric pairing on $TM \oplus T^*M$ and such that the space of sections of $\tilde{L}$, $\Gamma(\tilde{L})$, is closed under the Courant bracket $[,]^{\sim}$ on $\Gamma(TM \oplus T^*M)$ (see section 2.3, example 1). If $\tilde{L}$ is a Dirac structure on M, then $\tilde{L}$ is endowed with a Lie algebroid structure over M and the leaves of the induced Lie algebroid foliation $\mathcal{F}_{\tilde{L}}$ on M are presymplectic manifolds (see [2]). In the particular case when the Dirac structure $\tilde{L}$ comes from a Poisson structure Π on M, then $\tilde{L}$ is isomorphic to the cotangent Lie algebroid associated with Π and $\mathcal{F}_{\tilde{L}}$ is just the symplectic foliation of M (see [2]).

An algebraic treatment of Dirac structures was developed by Dorfman in [6] using the notion of a complex over a Lie algebra. This treatment was applied to the study of general Hamiltonian structures and their role in integrability. More recently, the properties of the

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Courant bracket $[,]^{\sim}$ have been systematized by Liu *et al* [23] in the definition of a Courant algebroid structure on a vector bundle $E \to M$ (see also [24, 30]). The natural example of a Courant algebroid is the Whitney sum $E = A \oplus A^*$, where the pair (A, A^*) is a Lie bialgebroid over M in the sense of Mackenzie–Xu [26].

On the other hand, a Jacobi structure on a manifold M is a local Lie algebra structure, in the sense of Kirillov [19], on the space $C^\infty(M, \mathbb{R})$ (see [4, 13, 22]; for an algebraic formulation of Jacobi structures, see [9]). We recall that a local Lie algebra structure on $C^\infty(M, \mathbb{R})$ is a Lie bracket which acts as a local operator on each of its arguments. Very recently, Grabowski and Marmo [12] proved that it is possible to skip the skew-symmetry assumption in the definition of a local Lie algebra (see also [10] for the particular case of a Poisson algebra). Apart from Poisson manifolds, interesting examples of Jacobi manifolds are contact and locally conformal symplectic manifolds. In fact, a Jacobi structure on M defines a generalized foliation, the characteristic foliation of M, whose leaves are contact or locally conformal symplectic manifolds [13, 19]. Moreover, the 1-jet bundle $T^*M \times \mathbb{R} \to M$ is a Lie algebroid and the corresponding Lie algebroid foliation is just the characteristic foliation of M (see [18]). However, for a Jacobi manifold M the vector bundle T^*M is not, in general, a Lie algebroid and, in addition, if one considers the usual Lie algebroid structure on $TM \times \mathbb{R}$ then the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not a Lie bialgebroid (see [16, 34]). Thus, it seems reasonable to introduce a proper definition of a Dirac structure on the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ (a $\mathcal{E}^1(M)$-Dirac structure in our terminology). This job was done by Wade in [35]. A $\mathcal{E}^1(M)$-Dirac structure is a vector sub-bundle L of $\mathcal{E}^1(M)$ that is maximally isotropic under the natural symmetric pairing of $\mathcal{E}^1(M)$ and such that the space $\Gamma(L)$ is closed under a suitable bracket $[,]$ on $\Gamma(\mathcal{E}^1(M))$ (this bracket may be defined using the general algebraic constructions of Dorfman [6]). Apart from $\mathcal{E}^1(M)$-Dirac structures which come from Dirac structures on M or from Jacobi structures, other examples can be obtained from a homogeneous Poisson structure on M, from a 1-form on M (a precontact structure in our terminology) or from a locally conformal presymplectic (lcp) structure, that is, a pair (Ω, ω), where Ω is a 2-form on M, ω is a closed 1-form and $\mathrm{d}\Omega = \omega \wedge \Omega$ (see [35]).

If L is a $\mathcal{E}^1(M)$-Dirac structure, $[,]_L$ is the restriction to $\Gamma(L) \times \Gamma(L)$ of the extended Courant bracket $[,]$ and ρ_L is the restriction to L of the canonical projection $\rho : \mathcal{E}^1(M) \to TM$, then the triple $(L, [,]_L, \rho_L)$ is a Lie algebroid over M (see [35]). Using the same terminology as in the Jacobi case, the Lie algebroid foliation $\mathcal{F}_L$ on M associated with L is called the characteristic foliation of L. An important remark is that the section ϕ_L of the dual bundle L^* defined by $\phi_L(e) = f$, for $e = (X, f) + (\alpha, g) \in \Gamma(L)$, is a 1-cocycle of the Lie algebroid $(L, [,]_L, \rho_L)$. Anyway, since $\mathcal{E}^1(M)$-Dirac structures are closely related with Jacobi structures, the presence of a Lie algebroid and a 1-cocycle in the theory is not very surprising (see [11, 15–17]).

Several aspects related to the geometry of $\mathcal{E}^1(M)$-Dirac structures were investigated by Wade in [35]. However, the nature of the induced structure on the leaves of the characteristic foliation of a $\mathcal{E}^1(M)$-Dirac structure L was not discussed in [35]. So, the aim of our paper is to describe such a nature. In addition, we will show that one may obtain, from L, a Dirac structure $\tilde{L}$ on $M \times \mathbb{R}$ in the sense of Courant–Weinstein and we will discuss the relation between the induced structures on the leaves of the characteristic foliations of L and $\tilde{L}$. For the above purposes, we will prove some general results about the relation between the 1-cocycles of an arbitrary Lie algebroid A over M and the leaves of the Lie algebroid foliation on M associated with A. In our opinion, these last results could be of independent interest.

The paper is organized as follows. In section 2, we recall several definitions and results about $\mathcal{E}^1(M)$-Dirac structures which will be used in the following. We also present some examples that were obtained in [35]. In section 3, we prove that if $(A, [\![,]\!], \rho)$ is a Lie algebroid over M, ϕ is a 1-cocycle of A and F is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ on M then $S_F^\phi = \emptyset$ or $S_F^\phi = F$, where S_F^ϕ is the subset of F defined by $S_F^\phi = \{x \in F / \ker(\rho_{|A_x}) \subseteq \langle\phi(x)\rangle^\circ\}$ (see theorem 3.2). Here, A_x is the fibre of A over x and $\langle\phi(x)\rangle^\circ$ is the annihilator of the subspace of A_x^* generated by $\phi(x)$. On the other hand, the Lie algebroid structure $([\![,]\!], \rho)$ and the 1-cocycle ϕ induce a Lie algebroid structure $([\![,]\!]^{-\phi}, \bar{\rho}^\phi)$ on the vector bundle $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ (see (3.1)). Then, if F and $\bar{F}$ are the leaves of the Lie algebroid foliations $\mathcal{F}_A$ and $\mathcal{F}_{\bar{A}}$ passing through $x_0 \in M$ and $(x_0, t_0) \in M \times \mathbb{R}$, we obtain, in the two possible cases $\left(S_F^\phi = \emptyset \text{ or } S_F^\phi = F\right)$, the relation between F and $\bar{F}$ (see theorem 3.3). Now, assume that L is a $\mathcal{E}^1(M)$-Dirac structure and that F is a leaf of the characteristic foliation $\mathcal{F}_L$. Then, in section 4, we prove that L induces, in a natural way, a $\mathcal{E}^1(F)$-Dirac structure L_F and, moreover (using the results of section 3), we describe the nature of L_F. In fact, we obtain that in the case when $S_F^{\phi_L} = \emptyset$, L_F comes from a precontact structure on F and in the case when $S_F^{\phi_L} = F$, L_F comes from a lcp structure on F (see theorem 4.2). Using this theorem, we directly deduce the results of Courant [2] about the leaves of the characteristic foliation of a Dirac structure and the results of Kirillov [19] and Guedira–Lichnerowicz [13] about the leaves of the characteristic foliation of a Jacobi structure. We also apply the theorem to the particular case when L comes from a homogeneous Poisson structure and some interesting consequences are derived. Finally, in section 5, we prove that a Dirac structure $\tilde{L}$ on $M \times \mathbb{R}$ can be obtained from a $\mathcal{E}^1(M)$-Dirac structure L in such a way that the Lie algebroid associated with $\tilde{L}$ is isomorphic to the Lie algebroid over $M \times \mathbb{R}$, $\left(\bar{L} = L \times \mathbb{R}, [,]_L^{-\phi_L}, \bar{\rho}_L^{\phi_L}\right)$. Thus, if (x_0, t_0) is a point of $M \times \mathbb{R}$, one may consider the leaves F and $\tilde{F}$ of the characteristic foliations of L and $\tilde{L}$ passing through x_0 and (x_0, t_0). Then, using the results of section 3, we obtain the relation between F and $\tilde{F}$ and, in addition, we describe the presymplectic 2-form on $\tilde{F}$ in terms of the precontact structure on F, when $S_F^{\phi_L} = \emptyset$, or in terms of the lcp structure on F, when $S_F^{\phi_L} = F$ (see theorem 5.3). As an application, we directly deduce some results of Guedira–Lichnerowicz [13] about the relation between the leaves of the characteristic foliation of a Jacobi structure on M and the leaves of the symplectic foliation of the Poisson structure on $M \times \mathbb{R}$ induced by the Jacobi structure.

2. $\mathcal{E}^1(M)$-Dirac structures

All the manifolds considered in this paper are assumed to be connected and of the class C^∞. Moreover, if M is a differentiable manifold, we will denote by $\mathcal{E}^1(M)$ the vector bundle $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \to M$. Note that the space of global sections $\Gamma(\mathcal{E}^1(M))$ of $\mathcal{E}^1(M)$ can be identified with the direct sum $(\mathfrak{X}(M) \times C^\infty(M, \mathbb{R})) \oplus (\Omega^1(M) \times C^\infty(M, \mathbb{R}))$.

2.1. *Definition and characterization of $\mathcal{E}^1(M)$-Dirac structures*

In this section, we will recall the definition of a $\mathcal{E}^1(M)$-Dirac structure, which was introduced by Wade in [35]. We will also give several results related to this notion.

The natural symmetric and skew-symmetric pairings $\langle,\rangle_+$ and $\langle,\rangle_-$ on $V \oplus V^*$, V being a real vector space of finite dimension, can be extended, in a natural way, to the Whitney sum $A \oplus A^*$, where $A \to M$ is a real vector bundle over a manifold M. We also denote by $\langle,\rangle_+$ and $\langle,\rangle_-$ the resultant pairings on $\Gamma(A \oplus A^*) \cong \Gamma(A) \oplus \Gamma(A^*)$. In the particular case when $A = TM \times \mathbb{R}$, the explicit expressions of $\langle,\rangle_+$ and $\langle,\rangle_-$ on $\Gamma(\mathcal{E}^1(M))$ are

$$\begin{aligned}\langle(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)\rangle_+ &= \tfrac{1}{2}\left(i_{X_2}\alpha_1 + f_2 g_1 + i_{X_1}\alpha_2 + f_1 g_2\right)\\ \langle(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)\rangle_- &= \tfrac{1}{2}\left(i_{X_2}\alpha_1 + f_2 g_1 - i_{X_1}\alpha_2 - f_1 g_2\right)\end{aligned} \tag{2.1}$$

for $(X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\mathcal{E}^1(M))$, $i \in \{1, 2\}$. One may also consider the homomorphism of $C^\infty(M, \mathbb{R})$-modules $\rho\colon \Gamma(\mathcal{E}^1(M)) \to \mathfrak{X}(M)$ defined by

$$\rho((X, f) + (\alpha, g)) = X. \tag{2.2}$$

On the other hand, in [35] Wade introduced a suitable $\mathbb{R}$-bilinear bracket $[,]\colon \Gamma(\mathcal{E}^1(M)) \times \Gamma(\mathcal{E}^1(M)) \to \Gamma(\mathcal{E}^1(M))$ on the space $\Gamma(\mathcal{E}^1(M))$. This approach is based on an idea that can be found in [6], where the author generalizes the Courant bracket on $\Gamma(TM \oplus T^*M)$ to the case of complexes over Lie algebras. The bracket [,] is given by

$$\begin{aligned}
[(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)] = {} & ([X_1, X_2], X_1(f_2) - X_2(f_1)) \\
& + \big(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \tfrac{1}{2}\mathrm{d}\left(i_{X_2}\alpha_1 - i_{X_1}\alpha_2\right) + f_1\alpha_2 - f_2\alpha_1 \\
& + \tfrac{1}{2}\left(g_2\,\mathrm{d}f_1 - g_1\,\mathrm{d}f_2 - f_1\,\mathrm{d}g_2 + f_2\,\mathrm{d}g_1\right), \\
& X_1(g_2) - X_2(g_1) + \tfrac{1}{2}\left(i_{X_2}\alpha_1 - i_{X_1}\alpha_2 - f_2 g_1 + f_1 g_2\right)\big)
\end{aligned} \tag{2.3}$$

for $(X_i, f_i)+(\alpha_i, g_i) \in \Gamma(\mathcal{E}^1(M))$, $i \in \{1, 2\}$, where [,] is the usual Lie bracket of vector fields and $\mathcal{L}$ is the Lie derivative operator on *M*. This bracket is skew-symmetric and, moreover, we have that

$$[e_1, f e_2] = f[e_1, e_2] + \rho(e_1)(f)e_2 - \langle e_1, e_2\rangle_+((0, 0) + (\mathrm{d}f, 0)) \tag{2.4}$$

for $e_1, e_2 \in \Gamma(\mathcal{E}^1(M))$ and $f \in C^\infty(M, \mathbb{R})$. We note that [,] is not, in general, a Lie bracket, since the Jacobi identity does not hold (see [35]).

Now, let *L* be a vector sub-bundle of $\mathcal{E}^1(M)$ which is isotropic under the symmetric pairing $\langle,\rangle_+$. We may consider the map $T_L : \Gamma(L) \times \Gamma(L) \times \Gamma(L) \to C^\infty(M, \mathbb{R})$ given by

$$T_L(e_1, e_2, e_3) = \langle [e_1, e_2], e_3\rangle_+ \qquad \text{for} \quad e_1, e_2, e_3 \in \Gamma(L). \tag{2.5}$$

If $e_i = (X_i, f_i) + (\alpha_i, g_i)$ with $i \in \{1, 2, 3\}$ then, using (2.1), (2.3), (2.5) and the fact that $\langle e_i, e_j\rangle_+ = 0$, for $i, j \in \{1, 2, 3\}$, we deduce that

$$\begin{aligned}
T_L(e_1, e_2, e_3) = \frac{1}{2} \sum_{Cycl.(e_1,e_2,e_3)} & \left(i_{[X_1,X_2]}\alpha_3 + g_3(X_1(f_2) - X_2(f_1)) + X_3\left(i_{X_2}\alpha_1 + f_2 g_1\right)\right. \\
& \left. + f_3\left(i_{X_2}\alpha_1 + f_2 g_1\right)\right).
\end{aligned} \tag{2.6}$$

Thus, from (2.4), (2.5) and (2.6), it follows that T_L is a skew-symmetric $C^\infty(M, \mathbb{R})$-trilinear map, that is, $T_L \in \Gamma(\wedge^3 L^*)$. Furthermore, using proposition 3.3 in [35], we obtain that

$$[[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2] = (0, 0) + (\mathrm{d}T_L(e_1, e_2, e_3), T_L(e_1, e_2, e_3)) \tag{2.7}$$

for $e_1, e_2, e_3 \in \Gamma(L)$.

Definition 2.1 [35]. *A $\mathcal{E}^1(M)$-Dirac structure on M is a sub-bundle L of $\mathcal{E}^1(M)$ which is maximally isotropic under the symmetric pairing $\langle,\rangle_+$ and such that $\Gamma(L)$ is closed under* [,].

It is clear that if *L* is a $\mathcal{E}^1(M)$-Dirac structure on *M* then the section $T_L \in \Gamma(\wedge^3 L^*)$ vanishes. In fact, we have the following result.

Proposition 2.2 [35]. *Let L be a sub-bundle of $\mathcal{E}^1(M)$ which is maximally isotropic under the symmetric pairing $\langle,\rangle_+$. Then, L is a $\mathcal{E}^1(M)$-Dirac structure if and only if the section $T_L \in \Gamma(\wedge^3 L^*)$ given by (2.5) vanishes.*

From (2.7) and proposition 2.2, we conclude that the restriction of [,] to $\Gamma(L)$ satisfies the Jacobi identity.

2.2. *Lie algebroids and the characteristic foliation of a $\mathcal{E}^1(M)$-Dirac structure*

A *Lie algebroid* A over a manifold M is a vector bundle A over M together with a Lie algebra structure $[\![,]\!]$ on the space $\Gamma(A)$ and a bundle map $\rho : A \to TM$, called the *anchor map*, such that if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M, \mathbb{R})$-modules induced by the anchor map then

(i) $\rho : (\Gamma(A), [\![,]\!]) \to (\mathfrak{X}(M), [,])$ is a Lie algebra homomorphism and
(ii) for all $f \in C^\infty(M, \mathbb{R})$ and for all $X, Y \in \Gamma(A)$, one has

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y.$$

The triple $(A, [\![,]\!], \rho)$ is called a Lie algebroid over M (see [25]).

Let $(A, [\![,]\!], \rho)$ be a Lie algebroid over M. We consider the generalized distribution $\mathcal{F}_A$ on M whose characteristic space at a point $x \in M$ is given by

$$\mathcal{F}_A(x) = \rho(A_x) \tag{2.8}$$

where A_x is the fibre of A over x. The distribution $\mathcal{F}_A$ is finitely generated and involutive. Thus, $\mathcal{F}_A$ defines a generalized foliation on M in the sense of Sussman [31]. $\mathcal{F}_A$ is the *Lie algebroid foliation* on M associated with A.

Remark 2.3. If F is the leaf of $\mathcal{F}_A$ passing through $x \in M$, $\dim F = r$ and $y \in M$ then $y \in F$ if and only if there exists a continuous piecewise smooth path $\gamma : I \to M$ from x to y, which is tangent to $\mathcal{F}_A$ and such that $\dim \mathcal{F}_A(\gamma(t)) = r$, for all $t \in I$ (see [20, 33]).

If $(A, [\![,]\!], \rho)$ is a Lie algebroid over M, one can introduce the *Lie algebroid cohomology complex with trivial coefficients* (see [25]). The space of 1-cochains is $\Gamma(A^*)$, where A^* is the dual bundle to A, and a 1-cochain $\phi \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi[\![X, Y]\!] = \rho(X)(\phi(Y)) - \rho(Y)(\phi(X)) \qquad \text{for all} \quad X, Y \in \Gamma(A). \tag{2.9}$$

Now, suppose that M is a differentiable manifold, that $[,]$ is the bracket on $\Gamma(\mathcal{E}^1(M))$ given by (2.3) and that $\rho : \Gamma(\mathcal{E}^1(M)) \to \mathfrak{X}(M)$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by (2.2). Also assume that L is a $\mathcal{E}^1(M)$-Dirac structure and that ρ_L (respectively, $[,]_L$) is the restriction of ρ (respectively, $[,]$) to $\Gamma(L)$ (respectively, $\Gamma(L) \times \Gamma(L)$). Then, it is clear that the triple $(L, [,]_L, \rho_L)$ is a Lie algebroid over M (see [35] and section 2.1). Thus, one can consider the Lie algebroid foliation $\mathcal{F}_L$ on M associated with L. $\mathcal{F}_L$ is called the *characteristic foliation of the $\mathcal{E}^1(M)$-Dirac structure*.

On the other hand, we may introduce a section ϕ_L of the dual bundle L^* as follows:

$$\phi_L(e) = f \qquad \text{for} \quad e = (X, f) + (\alpha, g) \in \Gamma(L). \tag{2.10}$$

A direct computation, using (2.2), (2.3) and (2.10), proves that ϕ_L is a 1-cocycle.

2.3. *Examples of $\mathcal{E}^1(M)$-Dirac structures*

Next, we will present some examples of $\mathcal{E}^1(M)$-Dirac structures which were obtained in [35]. In addition, we will describe the Lie algebroids, the characteristic foliations and the 1-cocycles associated with these structures.

2.3.1. Dirac structures. Let $\tilde{L}$ be a vector sub-bundle of $TM \oplus T^*M$ and consider the vector sub-bundle L of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma(L) = \{(X, 0) + (\alpha, f)/X + \alpha \in \Gamma(\tilde{L}),\ f \in C^\infty(M, \mathbb{R})\}. \tag{2.11}$$

Then, *$\tilde{L}$ is a Dirac structure on M* in the sense of Courant–Weinstein [2, 3] if and only if L is a $\mathcal{E}^1(M)$-Dirac structure (see [35]). We recall that a vector sub-bundle $\tilde{L}$ of $TM \oplus T^*M$ is a Dirac structure on M if $\tilde{L}$ is maximally isotropic under the natural symmetric pairing $\langle , \rangle_+$ on $TM \oplus T^*M$ and, in addition, the space of sections of $\tilde{L}$, $\Gamma(\tilde{L})$, is closed under the *Courant bracket* $[,]^\sim$ which is defined by

$$[X_1 + \alpha_1, X_2 + \alpha_2]^\sim = [X_1, X_2] + \left(\mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \tfrac{1}{2}\mathrm{d}\left(i_{X_2}\alpha_1 - i_{X_1}\alpha_2\right)\right) \tag{2.12}$$

for $X_1 + \alpha_1, X_2 + \alpha_2 \in \mathfrak{X}(M) \oplus \Omega^1(M) \cong \Gamma(TM \oplus T^*M)$.

If $\tilde{L} \subseteq TM \oplus T^*M$ is a Dirac structure on M then the triple $(\tilde{L}, [,]^\sim_{\tilde{L}}, \tilde{\rho}_{\tilde{L}})$ is a Lie algebroid over M, where $[,]^\sim_{\tilde{L}}$ is the restriction to $\Gamma(\tilde{L}) \times \Gamma(\tilde{L})$ of the Courant bracket given by (2.12) and $\tilde{\rho}_{\tilde{L}}$ is the restriction to $\Gamma(\tilde{L})$ of the map $\tilde{\rho} : \Gamma(TM \oplus T^*M) \to \mathfrak{X}(M)$ defined by

$$\tilde{\rho}(X + \alpha) = X \tag{2.13}$$

for all $X + \alpha \in \Gamma(TM \oplus T^*M)$ (see [2]). The *characteristic foliation associated with $\tilde{L}$* is the Lie algebroid foliation $\mathcal{F}_{\tilde{L}}$. It is clear that $\mathcal{F}_{\tilde{L}}(x) = \mathcal{F}_L(x)$, for all $x \in M$. In addition, from (2.10) and (2.11), it follows that the 1-cocycle ϕ_L identically vanishes.

2.3.2. Locally conformal presymplectic structures. A *locally conformal presymplectic (lcp) structure* on a manifold M is a pair (Ω, ω), where Ω is a 2-form on M, ω is a closed 1-form and $\mathrm{d}\Omega = \omega \wedge \Omega$. If (Ω, ω) is a lcp structure on M, one may define the vector sub-bundle $L_{(\Omega,\omega)}$ of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma\left(L_{(\Omega,\omega)}\right) = \left\{(X, -i_X\omega) + (i_X\Omega + f\omega, f)/(X, f) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})\right\}. \tag{2.14}$$

It is clear that the vector bundles $L_{(\Omega,\omega)}$ and $TM \times \mathbb{R}$ are isomorphic. In addition, $L_{(\Omega,\omega)}$ is a $\mathcal{E}^1(M)$-Dirac structure [35]. Note that if $\omega = 0$ then Ω is a *presymplectic form* on M. Furthermore, if (Ω, ω) is a lcp structure on a manifold M of even dimension and Ω is a nondegenerate 2-form then (Ω, ω) is a *locally conformal symplectic structure* (see [13, 19, 32]).

Let (Ω, ω) be a lcp structure on a manifold M and $L_{(\Omega,\omega)}$ be the associated $\mathcal{E}^1(M)$-Dirac structure. Then, using (2.2), (2.3) and (2.14), we deduce that the Lie algebroids $\left(L_{(\Omega,\omega)}, [,]_{L_{(\Omega,\omega)}}, \rho_{L_{(\Omega,\omega)}}\right)$ and $\left(TM \times \mathbb{R}, [\![,]\!]_{(\Omega,\omega)}, \rho_{(\Omega,\omega)}\right)$ are isomorphic, where $[\![,]\!]_{(\Omega,\omega)}$ and $\rho_{(\Omega,\omega)}$ are given by

$$[\![(X, f), (Y, g)]\!]_{(\Omega,\omega)} = ([X, Y], \Omega(X, Y) + (X(g) - g\omega(X)) - (Y(f) - f\omega(Y)))$$
$$\rho_{(\Omega,\omega)}(X, f) = X$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$. We remark that the map $\nabla : \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ defined by $\nabla_X f = X(f) - f\omega(X)$, for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M, \mathbb{R})$, induces a representation of the Lie algebroid $(TM, [,], \mathrm{Id})$ on the trivial vector bundle $M \times \mathbb{R} \to M$ and that Ω is a 2-cocycle of $(TM, [,], \mathrm{Id})$ with respect to this representation. In addition, the Lie algebroid $\left(TM \times \mathbb{R}, [\![,]\!]_{(\Omega,\omega)}, \rho_{(\Omega,\omega)}\right)$ is the extension of $(TM, [,], \mathrm{Id})$ via ∇ and Ω (for the definition of the extension of a Lie algebroid A with respect to a 2-cocycle and a representation of A on a vector bundle, see [25]). On the other hand, it is clear that $\mathcal{F}_{L_{(\Omega,\omega)}}(x) = T_xM$, for all $x \in M$, and that, under the isomorphism between $L_{(\Omega,\omega)}$ and $TM \times \mathbb{R}$, the 1-cocycle $\phi_{L_{(\Omega,\omega)}}$ is the pair $(-\omega, 0) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ (see (2.10) and (2.14)).

2.3.3. Precontact structures. A *precontact structure* on a manifold M is a 1-form η on M. A precontact structure η on M induces a $\mathcal{E}^1(M)$-Dirac structure L_η. More precisely, suppose that Φ is a 2-form on M, that η is a 1-form and consider the vector sub-bundle L of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma(L) = \left\{(X, f) + (i_X\Phi + f\eta, -i_X\eta)/(X, f) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})\right\}. \tag{2.15}$$

The vector bundles L and $TM \times \mathbb{R}$ are isomorphic. Moreover, L is a $\mathcal{E}^1(M)$-Dirac structure if and only if $\Phi = \mathrm{d}\eta$ (see [35]). Thus,

$$\Gamma(L_\eta) = \left\{(X, f) + (i_X\,\mathrm{d}\eta + f\eta, -i_X\eta)/(X, f) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})\right\}. \tag{2.16}$$

Note that a precontact structure η on a manifold M of odd dimension $2n+1$ such that $\eta \wedge (\mathrm{d}\eta)^n$ is a volume form is a *contact structure* (see [13, 19, 20, 22]).

Let η be a precontact structure on a manifold M and L_η be the associated $\mathcal{E}^1(M)$-Dirac structure. Then, the Lie algebroids $\left(L_\eta, [,]_{L_\eta}, \rho_{L_\eta}\right)$ and $(TM \times \mathbb{R}, [,], \pi)$ are isomorphic, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[,]$ is the usual Lie bracket on $\Gamma(TM \times \mathbb{R}) \cong \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ given by

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f))$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$. We also have that $\mathcal{F}_{L_\eta}(x) = T_xM$, for all $x \in M$. Moreover, under the isomorphism between L_η and $TM \times \mathbb{R}$, the 1-cocycle ϕ_{L_η} is the pair $(0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ (see (2.10) and (2.16)).

2.3.4. Jacobi structures. A *Jacobi structure* on a manifold M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field, such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$, $[,]$ being the Schouten–Nijenhuis bracket. If the vector field E identically vanishes then (M, Λ) is a *Poisson manifold*. Jacobi and Poisson manifolds were introduced by Lichnerowicz [21, 22] (see also [1, 4, 13, 19, 20, 33, 36]).

Now, given a 2-vector Λ and a vector field E on a manifold M, we can consider the vector sub-bundle $L_{(\Lambda,E)}$ of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma\left(L_{(\Lambda,E)}\right) = \left\{(\#_\Lambda(\alpha) + fE, -i_E\alpha) + (\alpha, f)/(\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})\right\} \tag{2.17}$$

where $\#_\Lambda : \Omega^1(M) \to \mathfrak{X}(M)$ is the homomorphism of $C^\infty(M, \mathbb{R})$-modules defined by $\beta(\#_\Lambda(\alpha)) = \Lambda(\alpha, \beta)$, for $\alpha, \beta \in \Omega^1(M)$. Note that the vector bundles $L_{(\Lambda,E)}$ and $T^*M \times \mathbb{R}$ are isomorphic. Moreover, we have that $L_{(\Lambda,E)}$ is a $\mathcal{E}^1(M)$-Dirac structure if and only if (Λ, E) is a Jacobi structure (see [35]).

If (Λ, E) is a Jacobi structure on a manifold M and $L_{(\Lambda,E)}$ is the associated $\mathcal{E}^1(M)$-Dirac structure then the Lie algebroids $\left(L_{(\Lambda,E)}, [,]_{L_{(\Lambda,E)}}, \rho_{L_{(\Lambda,E)}}\right)$ and $\left(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)}\right)$ are isomorphic, where $[\![,]\!]_{(\Lambda,E)}$ and $\tilde{\#}_{(\Lambda,E)}$ are defined by

$$\begin{aligned}[\![(\alpha, f), (\beta, g)]\!]_{(\Lambda,E)} = \big(&\mathcal{L}_{\#_\Lambda(\alpha)}\beta - \mathcal{L}_{\#_\Lambda(\beta)}\alpha - \mathrm{d}(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha \\ &- i_E(\alpha \wedge \beta), \Lambda(\beta, \alpha) + \#_\Lambda(\alpha)(g) - \#_\Lambda(\beta)(f) + fE(g) - gE(f)\big)\end{aligned}$$

$$\tilde{\#}_{(\Lambda,E)}(\alpha, f) = \#_\Lambda(\alpha) + fE$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ (see [35]). The Lie algebroid structure $\left([\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)}\right)$ on $T^*M \times \mathbb{R}$ was introduced in [18]. Recently, Grabowski and Marmo [11] defined the Poisson lift of the structure (Λ, E) and they proved that this lift defines, in a natural way, the Lie bracket $[\![,]\!]_{(\Lambda,E)}$. The characteristic foliation of $L_{(\Lambda,E)}$ is the characteristic foliation on M associated with the Jacobi structure (Λ, E) (see [4, 13, 19]) and, under the isomorphism between $L_{(\Lambda,E)}$ and $T^*M \times \mathbb{R}$, the 1-cocycle $\phi_{L_{(\Lambda,E)}}$ is the pair $(-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$ (see (2.10) and (2.17)).

2.3.5. Homogeneous Poisson structures. A *homogeneous Poisson manifold* (M, Π, Z) is a Poisson manifold (M, Π) with a vector field Z such that $[Z, \Pi] = -\Pi$ (see [4]). Given a 2-vector Π and a vector field Z on a manifold M, we can define the vector sub-bundle $L_{(\Pi,Z)}$ of $\mathcal{E}^1(M)$ whose sections are

$$\Gamma\big(L_{(\Pi,Z)}\big) = \big\{(\#_\Pi(\alpha) - fZ, f) + (\alpha, i_Z\alpha)/(\alpha, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})\big\}. \tag{2.18}$$

The vector bundles $L_{(\Pi,Z)}$ and $T^*M \times \mathbb{R}$ are isomorphic. Furthermore, (M, Π, Z) is a homogeneous Poisson manifold if and only if $L_{(\Pi,Z)}$ is a $\mathcal{E}^1(M)$-Dirac structure (see [35]).

Let (M, Π, Z) be a homogeneous Poisson manifold and $L_{(\Pi,Z)}$ be the associated $\mathcal{E}^1(M)$-Dirac structure. Then, from (2.2), (2.3) and (2.18), it follows that the Lie algebroids $\big(L_{(\Pi,Z)}, [,]_{L_{(\Pi,Z)}}, \rho_{L_{(\Pi,Z)}}\big)$ and $\big(T^*M \times \mathbb{R}, [\![,]\!]_{(\Pi,Z)}, \tilde{\#}_{(\Pi,Z)}\big)$ are isomorphic, where $[\![,]\!]_{(\Pi,Z)}$ and $\tilde{\#}_{(\Pi,Z)}$ are defined by

$$\begin{aligned}
[\![(\alpha, f), (\beta, g)]\!]_{(\Pi,Z)} = \big(&\mathcal{L}_{\#_\Pi(\alpha)}\beta - \mathcal{L}_{\#_\Pi(\beta)}\alpha - \mathrm{d}(\Pi(\alpha, \beta)) - f(\mathcal{L}_Z\beta - \beta) + g(\mathcal{L}_Z\alpha - \alpha),\\
&\#_\Pi(\alpha)(g) - \#_\Pi(\beta)(f) + gZ(f) - fZ(g)\big)
\end{aligned} \tag{2.19}$$

$$\tilde{\#}_{(\Pi,Z)}(\alpha, f) = \#_\Pi(\alpha) - fZ$$

for $(\alpha, f), (\beta, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$. Furthermore, it is clear that $\mathcal{F}_{L_{(\Pi,Z)}}(x) = \#_\Pi(T_x^*M) + \langle Z(x)\rangle$, for all $x \in M$. In other words, if $\mathcal{F}_\Pi$ is the symplectic foliation of the Poisson manifold (M, Π) then

$$\mathcal{F}_{L_{(\Pi,Z)}}(x) = \mathcal{F}_\Pi(x) + \langle Z(x)\rangle \qquad \text{for all} \quad x \in M. \tag{2.20}$$

In addition, under the isomorphism between $L_{(\Pi,Z)}$ and $T^*M \times \mathbb{R}$, the 1-cocycle $\phi_{L_{(\Pi,Z)}}$ is the pair $(0, 1) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$ (see (2.10) and (2.18)).

On the other hand, we may consider the Lie algebroid structure $([\![,]\!]_\Pi, \#_\Pi)$ on the vector bundle $T^*M \to M$ induced by the Poisson structure Π and the Lie algebroid structure $([\![,]\!]_Z, \rho_Z)$ on the vector bundle $M \times \mathbb{R} \to M$ induced by the vector field Z. The explicit definitions of $[\![,]\!]_\Pi$, $[\![,]\!]_Z$ and ρ_Z are

$$[\![\alpha, \beta]\!]_\Pi = \mathcal{L}_{\#_\Pi(\alpha)}\beta - \mathcal{L}_{\#_\Pi(\beta)}\alpha - \mathrm{d}(\Pi(\alpha, \beta))$$
$$[\![f, g]\!]_Z = gZ(f) - fZ(g) \qquad \rho_Z(f) = -fZ$$

for $\alpha, \beta \in \Omega^1(M)$ and $f, g \in C^\infty(M, \mathbb{R})$. Then, using (2.19), we conclude that $(T^*M, [\![,]\!]_\Pi, \#_\Pi)$ and $(M \times \mathbb{R}, [\![,]\!]_Z, \rho_Z)$ are Lie subalgebroids of $\big(T^*M \times \mathbb{R}, [\![,]\!]_{(\Pi,Z)}, \tilde{\#}_{(\Pi,Z)}\big)$. This implies that $(T^*M, [\![,]\!]_\Pi, \#_\Pi)$ and $(M \times \mathbb{R}, [\![,]\!]_Z, \rho_Z)$ form a matched pair of Lie algebroids in the sense of Mokri [27].

In section 4, we will prove that if F is a leaf of the characteristic foliation of a $\mathcal{E}^1(M)$-Dirac structure L then L induces a $\mathcal{E}^1(F)$-Dirac structure L_F and, in addition, we will describe the nature of L_F. First, in the next section, we will show two general results about the relation between the 1-cocycles of an arbitrary Lie algebroid A and the leaves of the Lie algebroid foliation $\mathcal{F}_A$.

3. 1-cocycles of a Lie algebroid and the leaves of the Lie algebroid foliation

Let $(A, [\![,]\!], \rho)$ be a Lie algebroid over M, $\phi \in \Gamma(A^*)$ be a 1-cocycle and $\pi_1 : M \times \mathbb{R} \to M$ be the canonical projection onto the first factor. We consider the map $\cdot : \Gamma(A) \times C^\infty(M \times \mathbb{R}, \mathbb{R}) \to C^\infty(M \times \mathbb{R}, \mathbb{R})$ given by

$$X \cdot \bar{f} = \rho(X)(\bar{f}) + \phi(X)\frac{\partial \bar{f}}{\partial t}.$$

It is easy to prove that $\cdot$ is an action of A on $M \times \mathbb{R}$ in the sense of [14] (see definition 2.3 in [14]). Thus, if π_1^*A is the pull-back of A over π_1 then the vector bundle $\pi_1^*A \to M \times \mathbb{R}$

admits a Lie algebroid structure $([\![,]\!]^{-\phi}, \bar{\rho}^{\phi})$ (see theorem 2.4 in [14]). For the sake of simplicity, when the 1-cocycle ϕ is zero, we will denote by $([\![,]\!]^{-}, \bar{\rho})$ the resultant Lie algebroid structure on $\pi_1^*A \to M \times \mathbb{R}$. On the other hand, it is clear that the vector bundles $\pi_1^*A \to M\times\mathbb{R}$ and $\bar{A} = A\times\mathbb{R} \to M\times\mathbb{R}$ are isomorphic and that the space of sections $\Gamma(\bar{A})$ of $\bar{A} \to M\times\mathbb{R}$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, we have that $[\![\bar{X}, \bar{Y}]\!]^{-}(x, t) = [\![\bar{X}_t, \bar{Y}_t]\!](x)$ and that $\bar{\rho}(\bar{X})(x, t) = \rho(\bar{X}_t)(x)$, for $\bar{X}, \bar{Y} \in \Gamma(\bar{A})$ and $(x, t) \in M \times \mathbb{R}$ (see [14]). In addition,

$$[\![\bar{X}, \bar{Y}]\!]^{-\phi} = [\![\bar{X}, \bar{Y}]\!]^{-} + \phi(\bar{X})\frac{\partial \bar{Y}}{\partial t} - \phi(\bar{Y})\frac{\partial \bar{X}}{\partial t} \qquad \bar{\rho}^{\phi}(\bar{X}) = \bar{\rho}(\bar{X}) + \phi(\bar{X})\frac{\partial}{\partial t} \tag{3.1}$$

where $\frac{\partial \bar{X}}{\partial t} \in \Gamma(\bar{A})$ denotes the derivative of $\bar{X}$ with respect to the time.

Now, if $\mathcal{F}_{\bar{A}}$ is the Lie algebroid foliation of $(\bar{A}, [\![,]\!]^{-\phi}, \bar{\rho}^{\phi})$ then, from (3.1), it follows that

$$\mathcal{F}_{\bar{A}}(x, t) = \left\{ \rho(e_x) + \phi(x)(e_x)\frac{\partial}{\partial t}_{|t} \in T_{(x,t)}(M \times \mathbb{R})/e_x \in A_x \right\} \tag{3.2}$$

for all $(x, t) \in M \times \mathbb{R}$. Moreover, a direct computation shows that

$$\dim \mathcal{F}_A(x) \leqslant \dim \mathcal{F}_{\bar{A}}(x, t) \leqslant \dim \mathcal{F}_A(x) + 1 \tag{3.3}$$

$$\dim \mathcal{F}_{\bar{A}}(x, t) = \dim \mathcal{F}_A(x) \qquad \Longleftrightarrow \qquad \ker(\rho_{|A_x}) \subseteq \langle \phi(x) \rangle^{\circ} \tag{3.4}$$

where $\mathcal{F}_A$ is the Lie algebroid foliation of A and $\langle \phi(x) \rangle^{\circ}$ is the annihilator of the subspace of A_x^* generated by $\phi(x)$, that is,

$$\langle \phi(x) \rangle^{\circ} = \{e_x \in A_x / \phi(x)(e_x) = 0\}.$$

Remark 3.1. Note that the vector field $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$ is an infinitesimal automorphism of the foliation $\mathcal{F}_{\bar{A}}$. Therefore, if (x, t_0) and (x, t_0') are points of $M \times \mathbb{R}$ and $\bar{F}, \bar{F}'$ are the leaves of $\mathcal{F}_{\bar{A}}$ passing through (x, t_0) and (x, t_0'), then the map $(y, s) \mapsto (y, s + (t_0' - t_0))$ is a diffeomorphism from $\bar{F}$ to $\bar{F}'$.

Next, we will discuss some relations between the leaves of $\mathcal{F}_A$ and the 1-cocycle ϕ and between the leaves of $\mathcal{F}_A$ and $\mathcal{F}_{\bar{A}}$. More precisely, the aim of this section is to prove the following two results.

Theorem 3.2. *Let $(A, [\![,]\!], \rho)$ be a Lie algebroid and $\phi \in \Gamma(A^*)$ be a 1-cocycle. If F is a leaf of the Lie algebroid foliation $\mathcal{F}_A$ and S_F^{ϕ} is the subset of F defined by*

$$S_F^{\phi} = \{x \in F/\ker(\rho_{|A_x}) \subseteq \langle \phi(x) \rangle^{\circ}\},$$

then $S_F^{\phi} = \emptyset$ or $S_F^{\phi} = F$. Furthermore, in the second case $\left(S_F^{\phi} = F\right)$, the 1-cocycle ϕ induces a closed 1-form ω_F on F which is characterized by the condition

$$\omega_F(\rho(e)_{|F}) = -\phi(e)_{|F} \qquad \text{for all} \quad e \in \Gamma(A). \tag{3.5}$$

Theorem 3.3. *Let $(A, [\![,]\!], \rho)$ be a Lie algebroid, $\phi \in \Gamma(A^*)$ be a 1-cocycle and consider on the vector bundle $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ the Lie algebroid structure $([\![,]\!]^{-\phi}, \bar{\rho}^{\phi})$ given by (3.1). Suppose that $(x_0, t_0) \in M \times \mathbb{R}$ and that F and $\bar{F}$ are the leaves of the Lie algebroid foliations $\mathcal{F}_A$ and $\mathcal{F}_{\bar{A}}$ passing through $x_0 \in M$ and $(x_0, t_0) \in M \times \mathbb{R}$, respectively. Then*

(i) if $\ker\left(\rho_{|A_{x_0}}\right) \not\subseteq \langle \phi(x_0) \rangle^{\circ}$ (or, equivalently, $S_F^{\phi} = \emptyset$) we have that $\bar{F} = F \times \mathbb{R}$.

(ii) if $\ker\left(\rho_{|A_{x_0}}\right) \subseteq \langle\phi(x_0)\rangle^\circ$ *(or, equivalently,* $S_F^\phi = F$*) and* $\pi_1 : M \times \mathbb{R} \to M$ *is the canonical projection onto the first factor, we have that* $\pi_1(\bar{F}) = F$ *and that the map* $\pi_{1|\bar{F}} : \bar{F} \to F$ *is a covering map. In addition, if* ω_F *is the closed 1-form on F characterized by condition (3.5) and* $i_{\bar{F}} : \bar{F} \to M \times \mathbb{R}$ *is the canonical inclusion then* $\bar{F}$ *is diffeomorphic to a Galois covering of F associated with* ω_F *and*

$$(\pi_{1|\bar{F}})^*(\omega_F) = -\mathrm{d}((i_{\bar{F}})^*(t)).$$

Proof of theorem 3.2. Let $\pi : A \to M$ be the canonical projection and denote by A_F the vector bundle over F defined by $A_F = \pi^{-1}(F)$. Using that F is a leaf of the Lie algebroid foliation on M associated with A, we obtain that the Lie algebroid structure on A induces a Lie algebroid structure $([\![,]\!]_F, \rho_F)$ on the vector bundle $\pi_F : A_F \to F$ and, in addition, the anchor map $\rho_F : A_F \to TF$ is an epimorphism of vector bundles. Thus, $(A_F, [\![,]\!]_F, \rho_F)$ is a transitive Lie algebroid. Denote by $K_F = \mathrm{Ker}\,(\rho_F)$ the *adjoint bundle* of A_F which is a bundle of Lie algebras over F (see [25], p 105).

Now, we consider the *adjoint representation* ad^F of A_F defined by

$$ad_e^F s = [\![e, s]\!] \tag{3.6}$$

for all $e \in \Gamma(A_F)$ and $s \in \Gamma(K_F)$ (see [25], p 107).

The flat A_F-connection ad^F on K_F induces, in a natural way, an A_F-connection on the dual bundle K_F^* to K_F, which we also denote by ad^F, and the restriction of the 1-cocycle ϕ to K_F defines a section ϕ^{K_F} of K_F^*. Furthermore, from (2.9) and (3.6), it follows that

$$\left(ad_e^F \phi^{K_F}\right)(s) = 0$$

for all $e \in \Gamma(A_F)$ and $s \in \Gamma(K_F)$, that is, ϕ^{K_F} is parallel with respect to ad^F.

Next, assume that $S_F^\phi \neq \emptyset$ and let x be a point of S_F^ϕ. This means that $\phi^{K_F}(x) = 0$. We must prove that $F = S_F^\phi$ or, equivalently, that if $y \in F$ and $s_y \in K_F(y) = \mathrm{Ker}\,(\rho_F(y))$,

$$\phi^{K_F}(y)(s_y) = 0.$$

For this purpose, we consider a continuous piecewise smooth path $\gamma : [0, 1] \to F$ from y to x. Then, using the results in [7], we deduce the following facts:

(i) There exists an A_F-path $\alpha : [0, 1] \to A_F$ with base path γ, i.e.

$$\rho_F \circ \alpha = \dot{\gamma}. \tag{3.7}$$

(ii) There exists a linear isomorphism $\tau_y^x : K_F(y) \to K_F(x)$, *the parallel displacement of the fibres along the* A_F*-path* α, which maps s_y on a point s_x of $K_F(x)$. In fact, if $\tilde{\gamma} : [0, 1] \to K_F$ is the unique horizontal lift of α (with respect to ad^F) starting at s_y then $\tau_y^x(s_y) = s_x = \tilde{\gamma}(1)$.

Now, choose a section e of A_F and a section s of K_F satisfying the following conditions

$$e \circ \gamma = \alpha \qquad s \circ \gamma = \tilde{\gamma}.$$

Then, (3.7) and the relations

$$0 = \left(ad_e^F s\right) \circ \gamma \qquad 0 = \left(ad_e^F \phi^{K_F}\right)(s) \circ \gamma$$

imply that the derivative of the map $\phi^{K_F} \circ \tilde{\gamma} : [0, 1] \to K_F \to \mathbb{R}$ is zero and thus

$$\phi^{K_F}(y)(s_y) = \phi^{K_F}(\gamma(0))(\tilde{\gamma}(0)) = \phi^{K_F}(\gamma(1))(\tilde{\gamma}(1)) = \phi^{K_F}(x)(s_x) = 0.$$

Finally, if $S_F^\phi = F$ we may introduce a 1-form ω_F on F given by

$$\omega_F(x)(\rho(e_x)) = -\phi(x)(e_x)$$

for all $x \in F$ and $e_x \in A_x$. Note that the condition

$$\ker(\rho_{|A_x}) \subseteq \langle \phi(x) \rangle^\circ \qquad \text{for all} \quad x \in F$$

implies that $\omega_F(x) : T_x F \to \mathbb{R}$ is well defined. Moreover, it is clear that ω_F satisfies (3.5) and, since ϕ is a 1-cocycle, we deduce that ω_F is closed. □

In order to prove theorem 3.3, we will use the following lemma.

Lemma 3.4. *If $\bar{F}$ is a leaf of the Lie algebroid foliation $\mathcal{F}_{\bar{A}}$ and $S^\phi_{\bar{F}}$ is the subset of $\bar{F}$ defined by*

$$S^\phi_{\bar{F}} = \left\{(x,t) \in \bar{F} / \ker\left(\rho_{|A_x}\right) \subseteq \langle \phi(x) \rangle^\circ \right\}$$

then $S^\phi_{\bar{F}} = \emptyset$ or $S^\phi_{\bar{F}} = \bar{F}$.

Proof. Assume that the dimension of $\bar{F}$ is r. Then, we will proceed in two stages. In the first stage, we will show that $S^\phi_{\bar{F}}$ is an open subset of $\bar{F}$. In the second stage, we will prove that $S^\phi_{\bar{F}}$ is a closed subset of $\bar{F}$. Thus, using the fact that $\bar{F}$ is connected, we will deduce that $S^\phi_{\bar{F}} = \emptyset$ or $S^\phi_{\bar{F}} = \bar{F}$.

First stage. Let (x_0, t_0) be a point of $S^\phi_{\bar{F}}$. We will show that there exists an open subset $\bar{W}_{(x_0,t_0)}$ of $\bar{F}$ such that $(x_0, t_0) \in \bar{W}_{(x_0,t_0)}$ and $\bar{W}_{(x_0,t_0)} \subseteq S^\phi_{\bar{F}}$ or, equivalently (see (3.4)),

$$\dim \mathcal{F}_A(x) = \dim \mathcal{F}_{\bar{A}}(x,t) = r \qquad \text{for all} \quad (x,t) \in \bar{W}_{(x_0,t_0)}. \tag{3.8}$$

Note that

$$\dim \mathcal{F}_{\bar{A}}(x,t) = r \qquad \text{for all} \quad (x,t) \in \bar{F}. \tag{3.9}$$

Therefore, we can choose a global generator system $\{e_1, \ldots, e_m\}$ of $\Gamma(A)$ in such a way that the set of vectors $\left\{\rho(e_i(x_0)) + \phi(x_0)(e_i(x_0)) \frac{\partial}{\partial t}_{|t_0}\right\}_{1 \leqslant i \leqslant r}$ is a basis of the vector space $\mathcal{F}_{\bar{A}}(x_0, t_0)$. Then, using that $(x_0, t_0) \in S^\phi_{\bar{F}}$, we deduce that the vectors $\{\rho(e_i(x_0))\}_{1 \leqslant i \leqslant r}$ are linearly independent in $T_{x_0}M$. This implies that $\dim \mathcal{F}_A(x_0) \geqslant r$ and, since the rank of a differentiable generalized distribution is a lower semicontinuous function (see [33]), there exists an open subset V'_{x_0} of M, $x_0 \in V'_{x_0}$, such that

$$\dim \mathcal{F}_A(x) \geqslant \dim \mathcal{F}_A(x_0) \geqslant r \qquad \text{for all} \quad x \in V'_{x_0}. \tag{3.10}$$

Thus, if $\bar{W}_{(x_0,t_0)}$ is the open subset of $\bar{F}$ defined by $\bar{W}_{(x_0,t_0)} = \bar{F} \cap \left(V'_{x_0} \times \mathbb{R}\right)$ then, from (3.3), (3.9) and (3.10), it follows that (3.8) holds.

Second stage. We will prove that $\bar{F} - S^\phi_{\bar{F}}$ is an open subset of $\bar{F}$.

Let (x_0, t_0) be a point of $\bar{F} - S^\phi_{\bar{F}}$ and suppose that F is the leaf of $\mathcal{F}_A$ passing through x_0. We have that $x_0 \in F - S^\phi_F$. Thus, using theorem 3.2, we deduce that

$$S^\phi_F = \emptyset. \tag{3.11}$$

Therefore, from (3.2) and (3.11), it follows that

$$\mathcal{F}_{\bar{A}}(x,t) = \mathcal{F}_A(x) \oplus \left\langle \frac{\partial}{\partial t}_{|t} \right\rangle \qquad \text{for all} \quad (x,t) \in F \times \mathbb{R}.$$

Consequently, $F \times \mathbb{R}$ is a connected integral submanifold of $\mathcal{F}_{\bar{A}}$ and its dimension is r. This implies that $F \times \mathbb{R}$ is an open subset of $\bar{F}$. Finally, from (3.11), we conclude that $F \times \mathbb{R} \subseteq \bar{F} - S^\phi_{\bar{F}}$. □

Proof of theorem 3.3. (i) If $\ker\left(\rho_{|A_{x_0}}\right) \not\subseteq \langle \phi(x_0) \rangle^\circ$ then $(x_0, t_0) \in \bar{F} - S^\phi_{\bar{F}}$ and thus, using lemma 3.4, we obtain that

$$S^\phi_{\bar{F}} = \emptyset. \tag{3.12}$$

Now, proceeding as in the second stage of the proof of lemma 3.4, we deduce that $F \times \mathbb{R}$ is an open subset of $\bar{F}$. On the other hand, from (3.3), (3.4) and (3.12), it follows that

$$\dim \mathcal{F}_A(x) = \dim \mathcal{F}_{\bar{A}}(x,t) - 1 = \dim \bar{F} - 1 \qquad \text{for all} \quad (x,t) \in \bar{F}.$$

Using this fact, (3.2) and remark 2.3, we have that $\pi_1(\bar{F}) \subseteq F$. Therefore, we have proved that $\bar{F} = F \times \mathbb{R}$.

(ii) Assume that $\ker\left(\rho_{|A_{x_0}}\right) \subseteq \langle \phi(x_0)\rangle^\circ$. Then, $(x_0, t_0) \in S^\phi_{\bar{F}}$ which implies that $\bar{F} = S^\phi_{\bar{F}}$, that is,

$$\dim \mathcal{F}_A(x) = \dim \mathcal{F}_{\bar{A}}(x,t) = \dim \bar{F} \qquad \text{for all} \quad (x,t) \in \bar{F}. \tag{3.13}$$

Using (3.2), (3.13) and remark 2.3, we obtain that $\pi_1(\bar{F}) \subseteq F$ and that $\pi_{1|\bar{F}} : \bar{F} \to F$ is a local diffeomorphism. Consequently, $\pi_1(\bar{F})$ is an open subset of F.

In addition, from (3.2) and (3.5), it follows that

$$(\pi_{1|\bar{F}})^*(\omega_F) = -\mathrm{d}((i_{\bar{F}})^* t). \tag{3.14}$$

Next, we will show that $\pi_1(\bar{F})$ is a closed subset of F and that $\pi_{1|\bar{F}} : \bar{F} \to F$ is a covering map.

Let x be a point of F. Since ω_F is a closed 1-form, there exists a connected open subset U in F and a real C^∞-differentiable function f_F on U such that $x \in U$ and

$$\omega_F = \mathrm{d} f_F \qquad \text{on} \quad U. \tag{3.15}$$

Then, using (3.2), (3.5), (3.15), remark 2.3 and the fact that $\pi_1(\bar{F}) \subseteq F$, we deduce the following result

$$(y,s) \in (\pi_{1|\bar{F}})^{-1}(U) = \pi_1^{-1}(U) \cap \bar{F} \quad \Rightarrow \quad \{(z, s + f_F(y) - f_F(z)) \in M \times \mathbb{R}/z \in U\} \subseteq \bar{F}. \tag{3.16}$$

Thus, if $x \in F - \pi_1(\bar{F})$, we have that $U \subseteq F - \pi_1(\bar{F})$. This proves that $\pi_1(\bar{F})$ is a closed subset of F which implies that $\pi_1(\bar{F}) = F$.

Now, suppose that (x,t) is a point of $\bar{F}$ and let U be a connected open subset of F and f_F be a real C^∞-differentiable function on F such that $x \in U$ and (3.15) holds. If $C_{(y,s)}$ is the connected component of a point $(y,s) \in (\pi_{1|\bar{F}})^{-1}(U)$ then, using (3.14) and (3.15), it follows that the function $(\pi_{1|\bar{F}})^*(f_F) + (i_{\bar{F}})^* t$ is constant on $C_{(y,s)}$. Therefore, from (3.16), we obtain that

$$C_{(y,s)} = \{(z, s + f_F(y) - f_F(z)) \in M \times \mathbb{R}/z \in U\}.$$

Consequently, the map $\pi_{1|C_{(y,s)}} : C_{(y,s)} \to U$ is a diffeomorphism. This proves that $\pi_{1|\bar{F}} : \bar{F} \to F$ is a covering map.

Finally, let E be the covering of F associated with ω_F, that is, E is the sheaf of germs of C^∞ functions g_F on F such that $\mathrm{d}g_F = \omega_F$ (see section 2 of chapter 14 in [8]). Denote by $\left(f^0_F\right)_{[x_0]}$ the germ of f^0_F at x_0, where f^0_F is a C^∞ function on a connected open subset U_0 of F such that $x_0 \in U_0$, $\left(f^0_F\right)(x_0) = t_0$ and $\omega_{F|U_0} = \mathrm{d} f^0_F$. Then, using the above description of the leaf $\bar{F}$ and the results in [8], we deduce that $\bar{F}$ is diffeomorphic to the connected component of $\left(f^0_F\right)_{[x_0]}$ in E. In other words, $\bar{F}$ is diffeomorphic to a Galois covering of F associated with ω_F. □

4. $\mathcal{E}^1(M)$-Dirac structures, submanifolds of the base space and the leaves of the characteristic foliation

4.1. $\mathcal{E}^1(M)$-Dirac structures and submanifolds of the base space

In this section, we will prove that if S is a submanifold of M then, under certain regularity conditions, a $\mathcal{E}^1(M)$-Dirac structure induces a $\mathcal{E}^1(S)$-Dirac structure. This result will be used in section 4.2.

Let L be a vector sub-bundle of $\mathcal{E}^1(M)$ which is maximally isotropic under the symmetric pairing $\langle,\rangle_+$ and S be a submanifold of M. If x is a point of S, we may define the vector space $(L_S)_x$ by

$$(L_S)_x = \frac{L_x \cap ((T_xS \times \mathbb{R}) \oplus (T_x^*M \times \mathbb{R}))}{L_x \cap (\{0\} \oplus ((T_xS)^\circ \times \{0\}))} \tag{4.1}$$

where $(T_xS)^\circ$ is the annihilator of T_xS, that is, $(T_xS)^\circ = \{\alpha \in T_x^*M/\alpha_{|T_xS} = 0\}$. We have that the linear map $(L_S)_x \to (T_xS \times \mathbb{R}) \oplus (T_x^*S \times \mathbb{R})$ given by

$$[(u,\lambda) + (\alpha,\mu)] \mapsto (u,\lambda) + (\alpha_{|T_xS}, \mu) \tag{4.2}$$

is a monomorphism and thus $(L_S)_x$ can be identified with a subspace of $(T_xS \times \mathbb{R}) \oplus (T_x^*S \times \mathbb{R})$. Moreover, using the results of section 1.4 in [2], we deduce that $(L_S)_x$ is a maximally isotropic subspace of $(T_xS \times \mathbb{R}) \oplus (T_x^*S \times \mathbb{R})$ under the symmetric pairing $\langle,\rangle_+$. In particular, this implies that $\dim (L_S)_x = \dim S + 1$, for all $x \in S$. In addition, we may prove the following proposition.

Proposition 4.1. *Let L be a $\mathcal{E}^1(M)$-Dirac structure and S be a submanifold of M. If the dimension of $L_x \cap ((T_xS \times \mathbb{R}) \oplus (T_x^*M \times \mathbb{R}))$ keeps constant for all $x \in S$ (or, equivalently, the dimension of $L_x \cap (\{0\} \oplus ((T_xS)^\circ \times \{0\}))$ keeps constant for all $x \in S$) then $L_S = \bigcup_{x\in S}(L_S)_x$ is a vector sub-bundle of $\mathcal{E}^1(S)$ and, furthermore, L_S is a $\mathcal{E}^1(S)$-Dirac structure.*

Proof. It is clear that L_S is a maximally isotropic vector sub-bundle of $\mathcal{E}^1(S)$ under the symmetric pairing $\langle,\rangle_+$.

Now, we consider the vector bundle $\hat{L}_S$ over S such that the fibre $(\hat{L}_S)_x$ of $\hat{L}_S$ over $x \in S$ is given by

$$(\hat{L}_S)_x = L_x \cap ((T_xS \times \mathbb{R}) \oplus (T_x^*M \times \mathbb{R})).$$

Denote by $i_S : \hat{L}_S \to L$ the inclusion map, by $\pi_S : \hat{L}_S \to L_S$ the canonical projection and by T_L (respectively, T_{L_S}) the section of $\wedge^3 L^*$ (respectively, $\wedge^3 L_S^*$) associated with the isotropic vector sub-bundle L (respectively, L_S). The map i_S (respectively, π_S) is a monomorphism (respectively, epimorphism) of vector bundles. Furthermore, if $e_i = (X_i, f_i) + (\alpha_i, g_i) \in \Gamma(\hat{L}_S)$, with $i \in \{1, 2, 3\}$, then, from (2.6) and proposition 2.2, we get that

$$\begin{aligned}\left(\pi_S^* T_{L_S}\right)(e_1, e_2, e_3) &= \frac{1}{2}\sum_{Cycl.(e_1,e_2,e_3)} \left(i_{[X_1,X_2]}\alpha_3 + g_3(X_1(f_2) - X_2(f_1))\right.\\ &\quad \left. + X_3\left(i_{X_2}\alpha_1 + f_2 g_1\right) + f_3\left(i_{X_2}\alpha_1 + f_2 g_1\right)\right)\\ &= (i_S^* T_L)(e_1, e_2, e_3) = 0.\end{aligned}$$

Therefore, $\pi_S^* T_{L_S} = 0$ and, since π_S is an epimorphism of vector bundles, we conclude that $T_{L_S} = 0$. This implies that L_S is a $\mathcal{E}^1(S)$-Dirac structure. □

4.2. The induced structure on the leaves of the characteristic foliation of a $\mathcal{E}^1(M)$-Dirac structure

Let L be a $\mathcal{E}^1(M)$-Dirac structure. Denote by $(L, [,]_L, \rho_L)$ the associated Lie algebroid and by $\phi_L \in \Gamma(L^*)$ the 1-cocycle defined by (2.10).

We consider the bundle map $(\rho_L, \phi_L) : L \to TM \times \mathbb{R}$ given by

$$(\rho_L, \phi_L)(e_x) = (\rho_L(e_x), \phi_L(x)(e_x)) \tag{4.3}$$

for $e_x \in L_x$ and $x \in M$. Then, we may define the 2-form $\Psi_L(x)$ on the vector space $(\rho_L, \phi_L)(L_x)$ by

$$\Psi_L(x)\left((\rho_L, \phi_L)((e_1)_x), (\rho_L, \phi_L)((e_2)_x)\right) = \langle (e_1)_x, (e_2)_x\rangle_-, \tag{4.4}$$

for $(e_1)_x, (e_2)_x \in L_x$, $\langle,\rangle_-$ being the natural skew-symmetric pairing on $(T_xM\times\mathbb{R})\oplus(T_x^*M\times\mathbb{R})$. Since L is a isotropic vector sub-bundle of $\mathcal{E}^1(M)$ under the symmetric pairing $\langle,\rangle_+$, we deduce that the 2-form $\Psi_L(x)$ is well defined. Note that if $e_1, e_2 \in \Gamma(L)$ then one may consider the function $\Psi_L((\rho_L,\phi_L)(e_1),(\rho_L,\phi_L)(e_2)) \in C^\infty(M,\mathbb{R})$ given by

$$\Psi_L((\rho_L,\phi_L)(e_1),(\rho_L,\phi_L)(e_2))(x) = \Psi_L(x)((\rho_L,\phi_L)((e_1)_x),(\rho_L,\phi_L)((e_2)_x)) \quad \text{for all} \quad x \in M.$$

In fact, if $e_i = (X_i, f_i) + (\alpha_i, g_i)$, with $i \in \{1, 2\}$, we have that

$$\Psi_L((X_1, f_1),(X_2, f_2)) = i_{X_2}\alpha_1 + f_2 g_1. \tag{4.5}$$

Now, let $\mathcal{F}_L$ be the characteristic foliation of the $\mathcal{E}^1(M)$-Dirac structure L and F be a leaf of $\mathcal{F}_L$. If x is a point of F, we will denote by $(L_F)_x$ the vector subspace of $(T_xF\times\mathbb{R})\oplus(T_x^*F\times\mathbb{R})$ given by (see section 4.1)

$$(L_F)_x = \frac{L_x \cap ((T_xF\times\mathbb{R})\oplus(T_x^*M\times\mathbb{R}))}{L_x\cap(\{0\}\oplus((T_xF)^\circ\times\{0\}))}.$$

Then, we will prove that $L_F = \bigcup_{x\in F}(L_F)_x$ defines a $\mathcal{E}^1(F)$-Dirac structure and we will describe the nature of L_F.

Theorem 4.2. *Let L be a $\mathcal{E}^1(M)$-Dirac structure and F be the leaf of the characteristic foliation $\mathcal{F}_L$ passing through $x_0 \in M$. Then, $L_F = \bigcup_{x\in F}(L_F)_x$ defines a $\mathcal{E}^1(F)$-Dirac structure and we have two possibilities:*

(i) If $\ker\left(\rho_{L|L_{x_0}}\right) \not\subseteq \langle\phi_L(x_0)\rangle^\circ$, the $\mathcal{E}^1(F)$-Dirac structure L_F comes from a precontact structure η_F on F, that is, $L_F = L_{\eta_F}$. In this case, F is said to be a precontact leaf.

(ii) If $\ker\left(\rho_{L|L_{x_0}}\right) \subseteq \langle\phi_L(x_0)\rangle^\circ$, the $\mathcal{E}^1(F)$-Dirac structure L_F comes from a lcp locally conformal presymplectic structure (Ω_F,ω_F) on F, that is, $L_F = L_{(\Omega_F,\omega_F)}$. In this case, F is said to be a lcp locally conformal presymplectic leaf.

Proof. From the definition of $\mathcal{F}_L$, it follows that

$$(\hat{L}_F)_x = L_x\cap((T_xF\times\mathbb{R})\oplus(T_x^*M\times\mathbb{R})) = L_x \qquad \text{for all} \quad x\in F.$$

Thus, since L is a maximally isotropic vector sub-bundle of $\mathcal{E}^1(M)$ under the symmetric pairing $\langle,\rangle_+$, we obtain that $\dim(\hat{L}_F)_x = \dim M + 1$, for all $x \in F$. Therefore, using proposition 4.1, we deduce that L_F defines a $\mathcal{E}^1(F)$-Dirac structure.

Next, we will distinguish the two cases:

(i) Assume that $\ker\left(\rho_{L|L_{x_0}}\right) \not\subseteq \langle\phi_L(x_0)\rangle^\circ$. Then, from theorem 3.2, we have that $\ker(\rho_{L|L_x}) \not\subseteq \langle\phi_L(x)\rangle^\circ$, for all $x \in F$. This implies that the map $(\rho_L,\phi_L)_{|L_x} : L_x \to T_xF\times\mathbb{R}$ is a linear epimorphism, for all $x\in F$ (see (4.3)). Consequently, the restriction of Ψ_L to F defines a section of the vector bundle $\wedge^2(T^*F\times\mathbb{R}) \to F$, i.e. a pair $(\Phi_F,\eta_F)\in\Omega^2(F)\times\Omega^1(F)$. The relation between Ψ_L and (Φ_F,η_F) is given by

$$\Psi_L(x)((u_1,\lambda_1),(u_2,\lambda_2)) = \Phi_F(x)(u_1,u_2) + \lambda_1\eta_F(x)(u_2) - \lambda_2\eta_F(x)(u_1) \tag{4.6}$$

for all $x\in F$ and $(u_1,\lambda_1),(u_2,\lambda_2)\in T_xF\times\mathbb{R}$.

Now, suppose that $(u,\lambda)+(\alpha,\mu)\in(\hat{L}_F)_x = L_x$, with $x\in F$. From (2.1), (4.4) and (4.6), it follows that

$$(i_u\Phi_F(x) + \lambda\eta_F(x))(v) + \nu(-\eta_F(x)(u)) = \alpha(v) + \nu\mu$$

for all $(v,\nu)\in T_xF\times\mathbb{R}$, that is, $\alpha_{|T_xF} = i_u\Phi_F(x) + \lambda\eta_F(x)$ and $\mu = -\eta_F(x)(u)$. In other words, if we consider $(L_F)_x$ to be a subspace of $(T_xF\times\mathbb{R})\oplus(T_x^*F\times\mathbb{R})$, we have that

$$(L_F)_x \subseteq \{(u,\lambda) + (i_u\Phi_F(x)+\lambda\eta_F(x), -\eta_F(x)(u))/(u,\lambda)\in T_xF\times\mathbb{R}\}.$$

But, since $\dim(L_F)_x = \dim F + 1$, we deduce that

$$(L_F)_x = \{(u, \lambda) + (i_u\Phi_F(x) + \lambda\eta_F(x), -\eta_F(x)(u))/(u, \lambda) \in T_xF \times \mathbb{R}\}.$$

Thus,

$$\Gamma(L_F) = \{(X, f) + (i_X\Phi_F + f\eta_F, -i_X\eta_F)/(X, f) \in \mathfrak{X}(F) \times C^\infty(F, \mathbb{R})\}. \quad (4.7)$$

Finally, using (4.7) and the fact that L_F is a $\mathcal{E}^1(F)$-Dirac structure, we conclude that (see section 2.3, example 3),

$$\Phi_F = \mathrm{d}\eta_F \quad (4.8)$$

and $L_F = L_{\eta_F}$.

(ii) Assume that $\ker\left(\rho_{L|L_{x_0}}\right) \subseteq \langle\phi_L(x_0)\rangle^\circ$. Then, from theorem 3.2, we obtain that $\ker(\rho_{L|L_x}) \subseteq \langle\phi_L(x)\rangle^\circ$, for all $x \in F$. This implies that the map

$$\varphi_x = \mathrm{pr}_{1|(\rho_L,\phi_L)(L_x)} : (\rho_L, \phi_L)(L_x) \to \rho_L(L_x) = T_xF$$

is a linear isomorphism, for all $x \in F$, where $\mathrm{pr}_1 : T_xF \times \mathbb{R} \to T_xF$ is the projection onto the first factor.

Therefore, Ψ_L induces a 2-form Ω_F on F which is characterized by the condition

$$\Omega_F(x)(\rho_L((e_1)_x), \rho_L((e_2)_x)) = \Psi_L(x)((\rho_L, \phi_L)((e_1)_x), (\rho_L, \phi_L)((e_2)_x)) \quad (4.9)$$

for $x \in F$ and $(e_1)_x, (e_2)_x \in L_x$. Moreover, since $S_F^{\phi_L} = F$, theorem 3.2 allows us to introduce the closed 1-form ω_F on F characterized by (3.5).

Now, suppose that $(u, \lambda) + (\alpha, \mu) \in (\hat{L}_F)_x = L_x$, with $x \in F$. From (2.1), (2.10), (3.5), (4.4) and (4.9), it follows that $\lambda = -\omega_F(x)(u)$ and that $\alpha_{|T_xF} = i_u\Omega_F(x) + \mu\omega_F(x)$. In other words, if we consider $(L_F)_x$ to be a subspace of $(T_xF \times \mathbb{R}) \oplus (T_x^*F \times \mathbb{R})$ then, since $\dim(L_F)_x = \dim F + 1$, we deduce that

$$(L_F)_x = \{(u, -\omega_F(x)(u)) + (i_u\Omega_F(x) + \mu\omega_F(x), \mu)/(u, \mu) \in T_xF \times \mathbb{R}\}.$$

Thus,

$$\Gamma(L_F) = \{(X, -\omega_F(X)) + (i_X\Omega_F + f\omega_F, f)/(X, f) \in \mathfrak{X}(F) \times C^\infty(F, \mathbb{R})\}. \quad (4.10)$$

Finally, using that ω_F is closed, (4.10) and the fact that L_F is a $\mathcal{E}^1(F)$-Dirac structure, we conclude that the pair (Ω_F, ω_F) is a lcp structure on F (see section 2.3, example 2) and that $L_F = L_{(\Omega_F,\omega_F)}$. □

Example 4.3

1. Dirac structures. Let $\tilde{L} \subseteq TM \oplus T^*M$ be a Dirac structure on M and L be the $\mathcal{E}^1(M)$-Dirac structure associated with $\tilde{L}$ (see (2.11)). We know that the characteristic foliations $\mathcal{F}_{\tilde{L}}$ and $\mathcal{F}_L$ associated with $\tilde{L}$ and L, respectively, coincide (see section 2.3, example 1). Thus, if $\tilde{F}$ is a leaf of $\mathcal{F}_{\tilde{L}}$ then, using theorem 4.2 and the fact that the 1-cocycle ϕ_L identically vanishes, it follows that $\tilde{F}$ carries an induced lcp structure $(\Omega_{\tilde{F}}, \omega_{\tilde{F}})$. Moreover, from the definition of $\omega_{\tilde{F}}$ (see (3.5)), we obtain that $\omega_{\tilde{F}} = 0$, that is, $\Omega_{\tilde{F}}$ is a presymplectic form on $\tilde{F}$. Therefore, we deduce a well-known result (see [2]): the leaves of the characteristic foliation $\mathcal{F}_{\tilde{L}}$ are presymplectic manifolds.

2. Locally conformal presymplectic structures. Let (Ω, ω) be a lcp structure on a manifold M and $L_{(\Omega,\omega)}$ be the corresponding $\mathcal{E}^1(M)$-Dirac structure (see (2.14)). It is clear that $\mathcal{F}_{L_{(\Omega,\omega)}}(x) = T_xM$, for all $x \in M$, and thus there is only one leaf of the foliation $\mathcal{F}_{L_{(\Omega,\omega)}}$, namely, M. Besides, since $\ker\left(\rho_{L_{(\Omega,\omega)}|(L_{(\Omega,\omega)})_x}\right) \subseteq \left\langle\phi_{L_{(\Omega,\omega)}}(x)\right\rangle^\circ$, for all $x \in M$ (see section 2.3, example 2), M carries an induced lcp structure which is just (Ω, ω).

3. Precontact structures. Let η be a precontact structure on a manifold M and denote by L_η the corresponding $\mathcal{E}^1(M)$-Dirac structure (see (2.16)). As in the case of a lcp structure, there is only one leaf of the characteristic foliation $\mathcal{F}_{L_\eta}$: the manifold M. In addition, since $\ker\left(\rho_{L_\eta|(L_\eta)_x}\right) \not\subseteq \langle\phi_{L_\eta}(x)\rangle^\circ$, for all $x \in M$ (see section 2.3, example 3), M carries an induced precontact structure. Such a structure is defined by the 1-form η.

4. Jacobi structures. Suppose that (Λ, E) is a Jacobi structure on a manifold M and let $L_{(\Lambda,E)}$ be the corresponding $\mathcal{E}^1(M)$-Dirac structure. We know that the characteristic foliation $\mathcal{F}_{L_{(\Lambda,E)}}$ of $L_{(\Lambda,E)}$ is just the characteristic foliation associated with the Jacobi structure (Λ, E) (see section 2.3, example 4). Moreover, using that the Lie algebroid $\left(L_{(\Lambda,E)}, [,]_{L_{(\Lambda,E)}}, \rho_{L_{(\Lambda,E)}}\right)$ can be identified with the Lie algebroid $\left(T^*M \times \mathbb{R}, [\![,]\!]_{(\Lambda,E)}, \tilde{\#}_{(\Lambda,E)}\right)$ and that, under this identification, the 1-cocycle $\phi_{L_{(\Lambda,E)}}$ is the pair $(-E, 0)$, we obtain that if x_0 is a point of M then

$$\ker\left(\rho_{L_{(\Lambda,E)}|\left(L_{(\Lambda,E)}\right)_{x_0}}\right) \subseteq \langle\phi_{L_{(\Lambda,E)}}(x_0)\rangle^\circ \quad\Longleftrightarrow\quad E(x_0) \in \#_\Lambda(T^*_{x0}M). \tag{4.11}$$

Thus, if F is the leaf of $\mathcal{F}_{L_{(\Lambda,E)}}$ passing through $x_0 \in M$ and $E(x_0) \in \#_\Lambda\left(T^*_{x_0}M\right)$, from (4.11) and theorem 4.2, it follows that F carries an induced lcp structure (Ω_F, ω_F). In fact, using (2.1), (3.5), (4.4) and (4.9), we have that

$$\Omega_F(y)(\#_\Lambda(\alpha_1), \#_\Lambda(\alpha_2)) = \alpha_1(\#_\Lambda(\alpha_2)) \qquad \omega_F(y)(\#_\Lambda(\alpha_1)) = \alpha_1(E(y))$$

for all $y \in F$ and $\alpha_1, \alpha_2 \in T^*_yM$. Therefore, the pair $(-\Omega_F, \omega_F)$ is the locally conformal symplectic structure on F induced by the Jacobi structure (Λ, E).

On the other hand, if F is the leaf of $\mathcal{F}_{L(\Lambda,E)}$ passing through $x_0 \in M$ and $E(x_0) \notin \#_\Lambda\left(T^*_{x_0}M\right)$ then, from (4.11) and theorem 4.2, we obtain that the $\mathcal{E}^1(F)$-Dirac structure comes from a precontact structure η_F on F. In addition, $E(y) \notin \#_\Lambda(T^*_yM)$ and $T_yF = \#_\Lambda(T^*_yM) \oplus \langle E(y)\rangle$, for all $y \in F$. Moreover, using (2.1), (4.4) and (4.6), we get that

$$\eta_F(y)(\#_\Lambda(\alpha) + \lambda E(y)) = -\lambda$$

for all $y \in F$, $\alpha \in T^*_yM$ and $\lambda \in \mathbb{R}$. Consequently, $-\eta_F$ is the contact structure on F induced by the Jacobi structure (Λ, E).

In conclusion, we deduce a well-known result (see [13, 19]): the leaves of the characteristic foliation of a Jacobi manifold are contact or locally conformal symplectic manifolds.

5. Homogeneous Poisson structures. Let (M, Π, Z) be a homogeneous Poisson manifold and $L_{(\Pi,Z)}$ be the corresponding $\mathcal{E}^1(M)$-Dirac structure (see (2.18)). Using that the Lie algebroid $\left(L_{(\Pi,Z)}, [,]_{L_{(\Pi,Z)}}, \rho_{L_{(\Pi,Z)}}\right)$ can be identified with the Lie algebroid $\left(T^*M \times \mathbb{R}, [\![,]\!]_{(\Pi,Z)}, \tilde{\#}_{(\Pi,Z)}\right)$ and that, under this identification, the 1-cocycle $\phi_{L_{(\Pi,Z)}}$ is the pair $(0, 1)$ (see section 2.3, example 5), we obtain that if x_0 is a point of M then

$$\ker\left(\rho_{L_{(\Pi,Z)}|\left(L_{(\Pi,Z)}\right)_{x_0}}\right) \subseteq \langle\phi_{L_{(\Pi,Z)}}(x_0)\rangle^\circ \quad\Longleftrightarrow\quad Z(x_0) \notin \#_\Pi\left(T^*_{x_0}M\right). \tag{4.12}$$

Thus, if x_0 is a point of M and F is the leaf of the characteristic foliation $\mathcal{F}_{L_{(\Pi,Z)}}$ passing through x_0, we will distinguish two cases:

(a) $Z(x_0) \in \#_\Pi\left(T^*_{x_0}M\right)$. In such a case, from (2.20), (4.12) and theorem 3.2, it follows that $T_yF = \mathcal{F}_{L_{(\Pi,Z)}}(y) = \mathcal{F}_\Pi(y)$, for all $y \in F$, where $\mathcal{F}_\Pi$ is the symplectic foliation of the Poisson manifold (M, Π). Therefore, F is the leaf of $\mathcal{F}_\Pi$ passing through x_0. In addition, using theorem 4.2, we deduce that the induced $\mathcal{E}^1(F)$-Dirac structure comes from a precontact structure η_F on F. Moreover, from (2.1), (4.4), (4.6) and (4.8), we have that

$$\eta_F(y)(\#_\Pi(\alpha_1)) = -\alpha_1(Z(y)) \qquad \mathrm{d}\eta_F(y)(\#_\Pi(\alpha_1), \#_\Pi(\alpha_2)) = \alpha_1(\#_\Pi(\alpha_2))$$

for all $y \in F$ and $\alpha_1, \alpha_2 \in T_y^*M$. This implies that $\mathrm{d}\eta_F$ is, up to sign, the symplectic 2-form of F.

(b) $Z(x_0) \notin \#_\Pi(T_{x_0}^*M)$. In such a case, from (2.20) and (4.12), we get that $T_yF = \mathcal{F}_{L_{(\Pi,Z)}}(y) = \mathcal{F}_\Pi(y) \oplus \langle Z(y)\rangle$, for all $y \in F$. Consequently, the dimension of F is odd and the leaf F_Π of the foliation $\mathcal{F}_\Pi$ passing through x_0 is a submanifold of F of codimension one. Furthermore, the induced $\mathcal{E}^1(F)$-Dirac structure comes from a lcp structure (Ω_F, ω_F) on F and, using (2.1), (3.5), (4.4) and (4.9), it follows that

$$\Omega_F(y)(\#_\Pi(\alpha_1) + \lambda_1 Z(y), \#_\Pi(\alpha_2) + \lambda_2 Z(y)) = \alpha_1(\#_\Pi(\alpha_2))$$
$$\omega_F(y)(\#_\Pi(\alpha_1) + \lambda_1 Z(y)) = \lambda_1$$

for all $y \in F, \alpha_1, \alpha_2 \in T_y^*M$ and $\lambda_1, \lambda_2 \in \mathbb{R}$. Note that if $i : F_\Pi \to F$ is the canonical inclusion, we deduce that $i^*\omega_F = 0$ and that $-i^*\Omega_F$ is the symplectic 2-form on F_Π. Thus, if the dimension of F is $2n+1$, we obtain that $\omega_F \wedge \Omega_F^n = \omega_F \wedge \Omega_F \wedge \overset{(n}{\cdots} \wedge \Omega_F$ is a volume form on F.

5. Dirac structure associated with a $\mathcal{E}^1(M)$-Dirac structure and characteristic foliations

Let M be a differentiable manifold and L be a vector sub-bundle of $\mathcal{E}^1(M)$.

We consider the vector sub-bundle $\tilde{L}$ of $T(M \times \mathbb{R}) \oplus T^*(M \times \mathbb{R})$ such that the fibre $\tilde{L}_{(x,t)}$ of $\tilde{L}$ over $(x,t) \in M \times \mathbb{R}$ is given by

$$\tilde{L}_{(x,t)} = \left\{ \left(u + \lambda \frac{\partial}{\partial t}_{|t} \right) + \mathrm{e}^t(\alpha + \mu\, \mathrm{d}t_{|t}) / (u, \lambda) + (\alpha, \mu) \in L_x \right\} \tag{5.1}$$

where L_x is the fibre of L over x. Note that the linear map $\psi_{(x,t)} : L_x \to \tilde{L}_{(x,t)}$ given by

$$\psi_{(x,t)}((u, \lambda) + (\alpha, \mu)) = \left(u + \lambda \frac{\partial}{\partial t}_{|t} \right) + \mathrm{e}^t(\alpha + \mu\, \mathrm{d}t_{|t}) \tag{5.2}$$

is an isomorphism of vector spaces, for all $(x,t) \in M \times \mathbb{R}$. Using this fact, (2.3) and (2.12), we deduce the following result.

Proposition 5.1. *L is a $\mathcal{E}^1(M)$-Dirac structure if and only if $\tilde{L}$ is a Dirac structure on $M \times \mathbb{R}$.*

Now, suppose that L is a $\mathcal{E}^1(M)$-Dirac structure and denote by $(L, [,]_L, \rho_L)$ the associated Lie algebroid and by ϕ_L the 1-cocycle of $(L, [,]_L, \rho_L)$ given by (2.10). Then, we may consider the Lie algebroid structure $\left([,]_L^{-\phi_L}, \bar{\rho}_L^{\phi_L}\right)$ defined by (3.1) on the vector bundle $\bar{L} = L \times \mathbb{R} \to M \times \mathbb{R}$.

On the other hand, let $\tilde{L}$ be the Dirac structure on $M \times \mathbb{R}$ associated with L, $(\tilde{L}, [,]\tilde{}_{\tilde{L}}, \tilde{\rho}_{\tilde{L}})$ be the corresponding Lie algebroid over $M \times \mathbb{R}$ and $\mathcal{F}_{\tilde{L}}$ be the characteristic foliation of $\tilde{L}$ (see section 2.3, example 1).

It is clear that the linear maps $\psi_{(x,t)}$, $(x,t) \in M \times \mathbb{R}$, induce an isomorphism of vector bundles $\tilde{\psi}$ between $\bar{L}$ and $\tilde{L}$. Moreover, we have

Lemma 5.2. *The map $\tilde{\psi}$ is an isomorphism of Lie algebroids over the identity, that is,*

$$\tilde{\rho}_{\tilde{L}}(\tilde{\psi}(\bar{e}_1)) = \bar{\rho}_L^{\phi_L}(\bar{e}_1) \qquad \tilde{\psi}[\bar{e}_1, \bar{e}_2]_L^{-\phi_L} = [\tilde{\psi}(\bar{e}_1), \tilde{\psi}(\bar{e}_2)]_{\tilde{L}}^{\sim} \tag{5.3}$$

for $\bar{e}_1, \bar{e}_2 \in \Gamma(\bar{L})$. Thus, the characteristic foliation $\mathcal{F}_{\tilde{L}}$ of the Dirac structure $\tilde{L}$ coincides with the Lie algebroid foliation $\mathcal{F}_{\bar{L}}$.

Proof. Using (2.2), (2.3), (2.10), (2.12), (2.13), (3.1) and (5.2), we deduce that (5.3) holds. In addition, from (5.3), it follows that $\mathcal{F}_{\tilde{L}}(x,t) = \mathcal{F}_{\bar{L}}(x,t)$, for all $(x,t) \in M \times \mathbb{R}$. □

Now, assume that $\tilde{F}$ is a leaf of the foliation $\mathcal{F}_{\tilde{L}} = \mathcal{F}_{\tilde{L}}$. Then, we know that $\tilde{F}$ is a presymplectic manifold with presymplectic 2-form $\Omega_{\tilde{F}}$ characterized by the condition

$$\Omega_{\tilde{F}}(x,t)\left(\tilde{\rho}_{\tilde{L}}\big((\tilde{e}_1)_{(x,t)}\big), \tilde{\rho}_{\tilde{L}}\big((\tilde{e}_2)_{(x,t)}\big)\right) = \big\langle (\tilde{e}_1)_{(x,t)}, (\tilde{e}_2)_{(X,t)}\big\rangle_- \tag{5.4}$$

for all $(x,t) \in M \times \mathbb{R}$ and $(\tilde{e}_1)_{(x,t)}, (\tilde{e}_2)_{(x,t)} \in \tilde{L}_{(x,t)}$, where $\langle , \rangle_-$ is the natural skew-symmetric pairing on $T_{(x,t)}(M\times\mathbb{R}) \oplus T^*_{(x,t)}(M\times\mathbb{R})$ (see [2] and examples 4.3).

Next, we will discuss the relation between the leaves of $\mathcal{F}_{\tilde{L}}$ and the leaves of the characteristic foliation $\mathcal{F}_L$ associated with L. In addition, we will describe the relation between the induced structures on them.

Theorem 5.3. *Let L be a $\mathcal{E}^1(M)$-Dirac structure and $\tilde{L}$ be the Dirac structure on $M\times\mathbb{R}$ associated with L. Suppose that $(x_0,t_0) \in M\times\mathbb{R}$ and that F and $\tilde{F}$ are the leaves of $\mathcal{F}_L$ and $\mathcal{F}_{\tilde{L}}$ passing through x_0 and (x_0,t_0), respectively. Then*
(i) if F is a precontact leaf we have that $\tilde{F} = F\times\mathbb{R}$. Moreover, if η_F is the precontact structure on F,

$$\Omega_{\tilde{F}} = \mathrm{e}^t\left((\pi_{1|\tilde{F}})^*(\mathrm{d}\eta_F) + \mathrm{d}t \wedge (\pi_{1|\tilde{F}})^*(\eta_F)\right)$$

where $\pi_{1|\tilde{F}} : \tilde{F} \to F$ is the restriction to $\tilde{F}$ of the canonical projection $\pi_1 : M\times\mathbb{R} \to M$.
(ii) if F is a lcp leaf and (Ω_F, ω_F) is the lcp structure on F then $\pi_1(\tilde{F}) = F$, $\pi_{1|\tilde{F}} : \tilde{F} \to F$ is a covering map and $\tilde{F}$ is diffeomorphic to a Galois covering of F associated with ω_F. Furthermore, if $i_{\tilde{F}} : \tilde{F} \to M\times\mathbb{R}$ is the canonical inclusion and $\tilde{\sigma} \in C^\infty(\tilde{F},\mathbb{R})$ is the function given by $\tilde{\sigma} = -(i_{\tilde{F}})^*(t)$, we have that

$$\mathrm{d}\tilde{\sigma} = (\pi_{1|\tilde{F}})^*(\omega_F) \qquad \Omega_{\tilde{F}} = \mathrm{e}^{-\tilde{\sigma}}(\pi_{1|\tilde{F}})^*(\Omega_F).$$

Proof. (i) Since F is a precontact leaf, it follows that $\ker\left(\rho_{L|L_{x_0}}\right) \not\subseteq \langle\phi_L(x_0)\rangle^\circ$ (see theorem 4.2). Thus, from theorem 3.3 and lemma 5.2, we deduce that $\tilde{F} = F\times\mathbb{R}$.

On the other hand, if $(x,t) \in \tilde{F}$, $(\tilde{e}_i)_{(x,t)} \in \tilde{L}_{(x,t)}, i \in \{1,2\}$, and $(\tilde{e}_i)_{(x,t)} = \left(u_i + \lambda_i \frac{\partial}{\partial t}_{|t}\right) + \mathrm{e}^t(\alpha_i + \mu_i\,\mathrm{d}t_{|t})$, with $(u_i,\lambda_i) + (\alpha_i,\mu_i) \in L_x$ then, using (4.4), (4.6), (4.8) and (5.4), we get

$$\begin{aligned}\Omega_{\tilde{F}}(x,t)\big(\tilde{\rho}_{\tilde{L}}((\tilde{e}_1)_{(x,t)}), \tilde{\rho}_{\tilde{L}}\big((\tilde{e}_2)_{(x,t)}\big)\big) &= \tfrac{1}{2}\,\mathrm{e}^t(\alpha_1(u_2) + \lambda_2\mu_1 - \alpha_2(u_1) - \mu_2\lambda_1)\\ &= \mathrm{e}^t((\pi_{1|\tilde{F}})^*(\mathrm{d}\eta_F) + \mathrm{d}t\wedge(\pi_{1|\tilde{F}})^*(\eta_F))(x,t)\big(\tilde{\rho}_{\tilde{L}}\big((\tilde{e}_1)_{(x,t)}\big), \tilde{\rho}_{\tilde{L}}\big((\tilde{e}_2)_{(x,t)}\big)\big).\end{aligned}$$

This implies that $\Omega_{\tilde{F}} = \mathrm{e}^t((\pi_{1|\tilde{F}})^*(\mathrm{d}\eta_F) + \mathrm{d}t\wedge(\pi_{1|\tilde{F}})^*(\eta_F))$.
(ii) If F is a lcp leaf then $\ker\left(\rho_{L|L_{x_0}}\right) \subseteq \langle\phi_L(x_0)\rangle^\circ$ (see theorem 4.2). Therefore, from theorem 3.3 and lemma 5.2, we obtain that $\pi_1(\tilde{F}) = F$, that $\pi_{1|\tilde{F}} : \tilde{F} \to F$ is a covering map, that $\tilde{F}$ is diffeomorphic to a Galois covering of F associated with ω_F and that $\mathrm{d}\tilde{\sigma} = (\pi_{1|\tilde{F}})^*(\omega_F)$.

Finally, if $(x,t) \in \tilde{F}$, $(\tilde{e}_i)_{(x,t)} \in \tilde{L}_{(x,t)}, i \in \{1,2\}$, and $(\tilde{e}_i)_{(x,t)} = \left(u_i + \lambda_i\frac{\partial}{\partial t}_{|t}\right) + \mathrm{e}^t(\alpha_i + \mu_i\,\mathrm{d}t_{|t})$, with $(u_i,\lambda_i) + (\alpha_i,\mu_i) \in L_x$ then, using (4.4), (4.9), (5.4) and the definition of $\tilde{\sigma}$, we deduce

$$\begin{aligned}\Omega_{\tilde{F}}(x,t)(\tilde{\rho}_{\tilde{L}}\left((\tilde{e}_1)_{(x,t)}\right), \tilde{\rho}_{\tilde{L}}\left((\tilde{e}_2)_{(x,t)}\right)) &= \tfrac{1}{2}\,\mathrm{e}^t(\alpha_1(u_2) + \lambda_2\mu_1 - \alpha_2(u_1) - \mu_2\lambda_1)\\ &= (\mathrm{e}^{-\tilde{\sigma}}(\pi_{1|\tilde{F}})^*(\Omega_F))(x,t)\left(\tilde{\rho}_{\tilde{L}}\left((\tilde{e}_1)_{(x,t)}\right), \tilde{\rho}_{\tilde{L}}\left((\tilde{e}_2)_{(x,t)}\right)\right).\end{aligned}$$

This implies that $\Omega_{\tilde{F}} = \mathrm{e}^{-\tilde{\sigma}}(\pi_{1|\tilde{F}})^*(\Omega_F)$. □

Example 5.4

1. Dirac structures. Let L be a $\mathcal{E}^1(M)$-Dirac structure which comes from a Dirac structure on M and $\tilde{L}$ be the associated Dirac structure on $M\times\mathbb{R}$. If x_0 is a point of M and F is the leaf of the characteristic foliation $\mathcal{F}_L$ passing through x_0, then F is a presymplectic manifold with presymplectic 2-form Ω_F (see examples 4.3). Moreover, since $\mathcal{F}_L(x) = \mathcal{F}_{\tilde{L}}(x, t)$, for all $(x, t) \in M \times \mathbb{R}$, we deduce that the leaf $\tilde{F}$ of $\mathcal{F}_{\tilde{L}}$ passing through $(x_0, t_0) \in M \times \mathbb{R}$ is $\tilde{F} = F \times \{t_0\}$. In addition, from theorem 5.3, it follows that $\Omega_{\tilde{F}} = \mathrm{e}^{t_0}\Omega_F$.

2. Precontact structures. Let η be a precontact structure on a manifold M and L_η be the associated $\mathcal{E}^1(M)$-Dirac structure. Then, the characteristic foliation $\mathcal{F}_{L_\eta}$ has a unique leaf, the manifold M (see examples 4.3). Furthermore, if $\tilde{L}_\eta$ is the Dirac structure on $M \times \mathbb{R}$ associated with L_η, we obtain that $\tilde{L}_\eta$ is the graph of the presymplectic 2-form $\tilde{\Omega}$ on $M \times \mathbb{R}$ given by

$$\tilde{\Omega} = \mathrm{e}^t((\pi_1)^*(\mathrm{d}\eta) + \mathrm{d}t \wedge (\pi_1)^*\eta).$$

In other words,

$$\Gamma(\tilde{L}_\eta) = \{\tilde{X} + i_{\tilde{X}}\tilde{\Omega}/\tilde{X} \in \mathfrak{X}(M \times \mathbb{R})\} \subseteq \mathfrak{X}(M \times \mathbb{R}) \oplus \Omega^1(M \times \mathbb{R}).$$

On the other hand, using theorem 5.3, we deduce a well-known result (see [2]): the characteristic foliation $\mathcal{F}_{\tilde{L}_\eta}$ of $\tilde{L}_\eta$ also has a unique leaf $\tilde{F}$ (the manifold $M \times \mathbb{R}$) and the presymplectic 2-form $\Omega_{\tilde{F}}$ on $\tilde{F}$ is just $\tilde{\Omega}$.

3. Jacobi structures. Suppose that (Λ, E) is a Jacobi structure on a manifold M. Then, it is well known that the 2-vector $\tilde{\Lambda}$ on $M \times \mathbb{R}$ given by $\tilde{\Lambda} = \mathrm{e}^{-t}\left(\Lambda + \frac{\partial}{\partial t} \wedge E\right)$ defines a Poisson structure on $M \times \mathbb{R}$ (see [22]; see also [4, 13, 29]). Thus, one may consider the Dirac structure $\tilde{L}_{\tilde{\Lambda}}$ on $M \times \mathbb{R}$ associated with $\tilde{\Lambda}$ (see [2]). In fact, we have that

$$\Gamma(\tilde{L}_{\tilde{\Lambda}}) = \{\#_{\tilde{\Lambda}}(\tilde{\alpha}) + \tilde{\alpha}/\tilde{\alpha} \in \Omega^1(M \times \mathbb{R})\}.$$

Moreover, if $L_{(\Lambda,E)}$ is the $\mathcal{E}^1(M)$-Dirac structure induced by the Jacobi structure (Λ, E), it is easy to prove that the Dirac structure $\tilde{L}_{(\Lambda,E)}$ on $M \times \mathbb{R}$ associated with $L_{(\Lambda,E)}$ is isomorphic to $\tilde{L}_{\tilde{\Lambda}}$. Therefore, using theorem 5.3 (see also examples 4.3), we directly deduce the results of Guedira–Lichnerowicz (see section 3.16 in [13]) about the relation between the leaves of the characteristic foliation of the Jacobi manifold (M, Λ, E) and the leaves of the symplectic foliation of the Poisson manifold $(M \times \mathbb{R}, \tilde{\Lambda})$.

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